Design of Optimal and Suboptimal Stability Augmentation Systems

C. A. Markland*
The Boeing Company, Seattle, Wash.

This paper is concerned with the application of optimal control techniques to the design of lateral-directional stability augmentation systems, and the use of optimal results in synthesizing practical, invariant, suboptimal systems. The main technique employed is implicit model-following. This utilizes an integral quadratic performance criterion in which the difference between the dynamics of the plant and those of an ideal or model system are minimized. The specification of ideal airplane dynamics is discussed in terms of satisfying typical handling qualities requirements, and a direct method for selecting the performance criterion weighting factors is employed. The problem of measuring state variables is alleviated by means of suitable transformations. A simpler type of optimization involving the generalized inverse matrix is presented, and some interesting analytical results are obtained which show the relationship between modern methods and those used previously. Suboptimal systems based on the optimal results are studied, and it is shown that even for supersonic aircraft it is possible in some cases to achieve satisfactory handling qualities with an almost invariant stability augmentation system.

Introduction

SINCE the original work of Kalman, Merriam, and others in the theory of optimal control, and particularly the solution for the case of a linear plant subject to a quadratic performance criterion[1,2] the application of this theory to the lateral-directional control of aircraft has become a popular example in the literature.[3-6] It is indeed a very good example since the equations of motion are well defined, and classical control system synthesis techniques are not directly applicable to this multivariable type of problem. In the optimal control approach multivariable systems are considered naturally through the state variable formulation.

In spite of the many published applications, many practicing engineers have had reservations about the ultimate value of the theory in the real situation. The objections most frequently center around the many feedback loops inherent in the optimal solution and the difficulty of measuring all the state variables. However, it can be shown that by the use of suitable transformations, unmeasurable variables can be replaced by other variables which can easily be sensed. Furthermore, good suboptimal systems can be obtained from the optimal results by utilizing simple filters and with several of the feedback loops omitted.

Another controversial point, and a fundamental requirement of any real optimization problem, is the mathematical definition of optimum performance. In many problems this is obvious, for example in minimum time or minimum fuel problems, but in the case of the airplane lateral-directional control problem it is less definite. However, a reasonable objective is that the optimal controller should produce a response as close as possible to that of a prescribed ideal or model system using a minimum of control surface activity. This objective can be formulated in terms of an integral quadratic performance criterion in two distinct ways, namely real and implicit model-following, but the engineer must also specify numerical values for the weighting factors to indicate the relative importance of the individual terms. Even when the system has been optimized according to the criterion, a multitude of other important aspects must be considered, for example cost of implementation, effect of failures and interaction with nonlinearities and higher order dynamics. It is not possible to include all of these factors in the criterion of optimality so that it would be unrealistic to suppose that optimization automatically provides satisfaction, but there is no reason to believe that optimal control theory is more sensitive to these problems than other design techniques. In fact, current experience has shown that very promising and practical systems can result from the optimal control approach.

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* Research Engineer, Commercial Airplane Division; now with the European Space Research Organization, Noordwijk, Holland.

Theory of Optimal Model-Following

Consider a linear plant described in state variable form by

$$\dot{\mathbf{x}}_p = A\mathbf{x}_p + B\mathbf{u} \tag{1}$$

where $\mathbf{x}_p$ is the vector of plant state variables and $\mathbf{u}$ is the control vector. It is postulated that the objective is to employ control in such a way that the closed loop response follows the response of a model or "ideal" system defined by

$$\dot{\mathbf{x}}_m = H\mathbf{x}_m \tag{2}$$

A quadratic performance criterion appropriate to this problem is[3]

$$\min_{\mathbf{u}} \int_{t_0}^{t_f} \{(\mathbf{x}_p - \mathbf{x}_m)'Q(\mathbf{x}_p - \mathbf{x}_m) + \mathbf{u}'R\mathbf{u}\}dt \tag{3}$$

Here differences between plant and model response are quadratically penalized along with control effort. Including $\mathbf{x}_m$ in the performance criterion requires that it must also be incorporated into an augmented set of system equations

$$\begin{bmatrix}\dot{\mathbf{x}}_p\\ \dot{\mathbf{x}}_m\end{bmatrix} = \begin{bmatrix}A, & 0\\ 0, & H\end{bmatrix}\begin{bmatrix}\mathbf{x}_p\\ \mathbf{x}_m\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}\mathbf{u} \tag{4}$$

and further these variables must be measured by the control system. Accordingly, the model must be synthesized as an actual part of the controller (a prefilter), and this technique has been termed real model-following.

A quite different approach to the same problem is to minimize the difference between the dynamics of the actual and the model systems, i.e., use the performance criterion

$$\min_{\mathbf{u}} \int_{t_0}^{t_f} \{(\dot{\mathbf{x}}_p - \dot{\mathbf{x}}_m)'Q(\dot{\mathbf{x}}_p - \dot{\mathbf{x}}_m) + \mathbf{u}'R\mathbf{u}\}dt =$$

$$\min_{\mathbf{u}} \int_{t_0}^{t_f} \{(A\mathbf{x} + B\mathbf{u} - H\mathbf{x})'Q(A\mathbf{x} + B\mathbf{u} - H\mathbf{x}) + \mathbf{u}'R\mathbf{u}\}dt \tag{5}$$

Clearly, reducing the deviation $(A\mathbf{x} + B\mathbf{u} - H\mathbf{x})$ implies that the plant is following the model dynamics although it is not following a physical model as in the previous formulation. Hence, this is referred to as implicit model-following.

The solution of these optimal control problems can be found by classical calculus of variations or the dynamic programing approach. Both require the solution of a matrix ricatti equation, but the implicit model-following case is complicated slightly by the presence of products of $\mathbf{x}$ and $\mathbf{u}$ in the performance index. For this case the optimal control law is

$$\mathbf{u}^0 = -\hat{R}^{-1}[B'Q(A - H) + B'P]\mathbf{x} = D\mathbf{x} \tag{6}$$

where matrix P is the solution of the ricatti equation

$$\dot{P} + \hat{Q} + P\hat{A} + \hat{A}'P - PB\hat{R}^{-1}B'P = 0, P(t_f) = 0 \tag{7}$$

and

$$R = R + B'QB$$

$$\hat{Q} = (A - H)'(Q - QB\hat{R}^{-1}B'Q)(A - H)$$

$$\hat{A} = A - B\hat{R}^{-1}B'Q(A - H)$$

To briefly compare real and implicit model-following: the primary difference is that implicit model-following is essentially an attempt to match the plant and model dynamics, whereas in real model-following feedback is employed merely to speed up the plant response so that the over-all response tends toward the model response. In some control problems perfect matching is a practical solution and plainly the latter technique is then inappropriate. Furthermore, the necessity of having a fast plant response involves high-feedback gains that cannot be allowed in many instances due to stability problems with higher order dynamics and nonlinearities not included in the plant equations. On the other hand, because real model-following does use high gains and a built-in model it is less sensitive to changes in plant parameters. However, for the synthesis of stability augmentation systems the high gains and the complication of obtaining the response of a varying model to external disturbances make real model-following impractical in all but extreme cases. Therefore, only implicit model-following will be considered further.

Equations of Motion

The specific control problem considered here is the control of the lateral-directional motions of aircraft. The linearized equations of motion are in state variable form

$$\begin{bmatrix} \dot{\beta} \\ \dot{\phi} \\ \dot{P}_s \\ \dot{R}_s \end{bmatrix} = \begin{bmatrix} Y_v & Y_\phi & Y_p^* & Y_r^*-1 \\ 0 & 0 & 1 & 0 \\ L'_\beta & 0 & L'_p & L'_r \\ N'_\beta & 0 & N'_p & N'_r \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ P_s \\ R_s \end{bmatrix} + \begin{bmatrix} Y_{\delta_R}^* & Y_{\delta_A}^* \\ 0 & 0 \\ L'_{\delta_R} & L'_{\delta_A} \\ N'_{\delta_R} & N'_{\delta_A} \end{bmatrix} \begin{bmatrix} \delta_R \\ \delta_A \end{bmatrix} \tag{8}$$

where the matrices contain the star and primed aerodynamic derivatives used in the standard literature,[7,8] β is the sideslip angle, ϕ is the eulerian bank angle (closely approximated by the integral of P_s), P_s and R_s are roll and yaw rates about the stability axes, and δ_R and δ_A are the rudder and aileron surface deflections respectively. Since the over-all control system must also respond to pilot commands from wheel and pedals, these inputs are appended to the basic airplane equations as uncontrollable state variables and they are assumed to have steady values. Therefore,

$$\dot{\delta}_w = 0 \text{ and } \dot{\delta}_p = 0 \tag{9}$$

add a fifth and sixth row and column of zeros to the matrices in Eq. (8). The fact that the system is not now completely controllable means that a steady-state solution to the ricatti equation may not exist. However, if the preceding steady input equations are modified to give very small but negative eigenvalues (e.g., $\dot{\delta}_w = -0.001\delta_w$) then the over-all system is stabilizable and this problem does not arise.

Specification of Model Dynamics

The model or ideal system dynamics are defined by the elements of matrix H in Eq. (2), and this is formed by considering the equations of motion of the basic airplane, viz.,

$$\begin{bmatrix} \dot{\beta} \\ \dot{\phi} \\ \dot{P}_s \\ \dot{R}_s \\ \dot{\delta}_w \\ \dot{\delta}_p \end{bmatrix} = \begin{bmatrix} Y_v & Y_\phi & Y_p^* & Y_r^* - 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ L'_\beta & 0 & L'_p & L'_r & 0 & 0 \\ N'_\beta & 0 & N'_p & N'_r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ P_s \\ R_s \\ \delta_w \\ \delta_p \end{bmatrix} \tag{10}$$

The derivatives Y_p^*, Y_r^*, L'_r, and N'_p constitute undesirable aerodynamic coupling between roll, yaw, and sideslip motions so they are set equal to zero in the model. The term L'_β represents the roll moment due to sideslip or dihedral effect arising from wing dihedral and sweepback. Positive dihedral is desirable to provide static stability, but in some flight conditions it can be excessive, and therefore in the model the basic value of L'_β is attenuated by a parameter $k_\beta \leq 1$. The desired response to pilot wheel commands is specified by setting $H_{35} = -(P/\delta_w)L'_p$, so that wheel produces only roll directly. Pedal inputs are accommodated by observing that the pilot uses pedals primarily to produce or counteract sideslip for example in a cross-wind landing. Assuming that he uses wheel to keep the wings level this condition requires

$$\dot{\beta} = \dot{P} = \dot{R} = P = R = \phi = 0 \tag{11}$$

therefore

$$\dot{\beta} = 0 = Y_v\beta + H_{16}\delta_p, \quad \dot{R} = 0 = N'_\beta\beta + H_{46}\delta_p \tag{12}$$

defining matrix elements $H_{16} = -Y_v(\beta/\delta_p)$ and $H_{46} = -N'_\beta(\beta/\delta_p)$ where (β/δ_p) is the required pedal sensitivity. Now the model equations are

$$\begin{aligned} \dot{\beta} &= Y_v\beta + Y_\phi\phi - R_s - Y_v(\beta/\delta_p)\delta_p \\ \dot{P}_s &= k_\beta L'_\beta\beta + L'_pP_s - L'_p(P/\delta_w)\delta_w \\ \dot{R}_s &= N'_\beta\beta + N'_rR_s - N'_\beta(\beta/\delta_p)\delta_p \end{aligned} \tag{13}$$

and it is noted that it is not possible to do a coordinated turn without pilot inputs with this system; i.e., for nonzero ϕ and R_s

$$\dot{\beta} = \dot{P} = \dot{R} = \beta = P = \delta_w = \delta_p = 0$$

is not a solution to these equations. This condition is corrected, and neutral spiral stability is ensured, by adding the term $-N'_rY_\phi\Phi$ to the yaw acceleration equation. Hence the final model system is

$$\begin{bmatrix} \dot\beta \\ \dot\phi \\ \dot P_s \\ \dot R_s \\ \dot\delta_w \\ \dot\delta_p \end{bmatrix} = \begin{bmatrix} Y_v & Y_\phi & 0 & -1 & 0 & -Y_v(\beta/\delta_p) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ k_\beta L'_\beta & 0 & L'_p & 0 & -L'_p(P/\delta_w) & 0 \\ N_\beta' & -N'_r Y_\phi & 0 & N'_r & 0 & -N'_\beta(\beta/\delta_p) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ P_s \\ R_s \\ \delta_w \\ \delta_p \end{bmatrix} \quad (14)$$

In order to assign numerical values to the elements of this coefficient matrix consider the location of its eigenvalues. The characteristic equation

$$\det (\lambda I - H) = 0$$

can be approximately factored to

$$\lambda^3(\lambda - L'_p)[\lambda^2 + (-Y_v - N'_r)\lambda + (N'_\beta + Y_v N'_r + Y_\phi k_\beta L'_\beta/L'_p)] = 0 \quad (15)$$

(This has been found accurate to within 2% in most cases.) The three zero roots correspond to the wheel and pedal steps and the neutral spiral mode, the real root $\lambda = L'_p$ is the roll mode, and the quadratic term represents the dutch roll oscillation.

Aircraft performance specifications and other handling qualities references provide data on appropriate values for these roots. Typical requirements on roll time constant are 0.5–1.0 sec, and for the model the lower end of this range was chosen to be nearer to the fast response with little bank angle overshoot preferred by test pilots. Hence, $L'_p = H_{33} \leq -2.0$. Dutch roll requirements specify Y_v, N'_β and N'_r but they are usually rather vague. Some criteria show lower bounds of 0.7–1.0 rad/sec on dutch roll frequency and 0.08 on damping, while others may specify that the dutch roll mode must decay to half amplitude in less than say 5 sec. Ashkenas[9] concluded from a survey of many aircraft that satisfactory pilot ratings are obtained for $(\xi\omega)_{DR}$ in the range 0.2–0.3, corresponding to $T_{1/2}$ between 3.5 and 2.3 sec, for frequencies between 0.8 and 6.0 rad/sec and for aircraft with little dihedral effect. Accordingly, the specification for the model, ideal airplane, system was set at time-to-half-amplitude of 2.0 sec, or $(\xi\omega)_{DR} = 0.35$, to allow for the residual dihedral and imperfect matching, and have damping greater than 0.15 and frequency greater than 0.8 rad/sec. Relating this information to the elements of H and Eq. (15)

$$-(Y_v + N'_r) = 2\xi\omega, \quad N'_\beta + Y_v N'_r + Y_\phi k_\beta L'_\beta/L'_p = \omega^2 \quad (16)$$

so the yaw damping derivative is first computed by

$$N'_r = -Y_v - 2\xi\omega$$

and then the directional stability is augmented according to

$$N'_\beta = \omega^2 - Y_v N'_r - Y_\phi k_\beta L'_\beta/L'_p$$

It is postulated that Y_v should be left at its basic airplane value, and that the model dutch roll frequency should be increased by a factor $k_\omega \geq 1.0$ over the unaugmented value and be not less than 0.8 rad/sec.

The choice of parameters k_β and k_ω can only be made on the basis of experience with particular aircraft. In practice, k_β has been adjusted so that the roll rates experienced in side gusts are moderate, and the dutch roll frequency has been augmented to satisfy turn entry and engine failure transient criteria and flexibility restrictions.

The wheel and pedal sensitivities will depend on the type of aircraft but will generally be specified or implied in performance requirements. For a transport airplane, typically $(P/\delta_w) = 0.45°$/sec per degree of wheel and (β/δ_p) may be set to counteract the sideslip in a maximum cross-wind landing condition.

Specification of the Performance Criterion

The performance criterion for implicit model-following has the general quadratic form

$$\min_{\mathbf{u}} \int_{t_0}^{t_f} \{(\dot{\mathbf{x}}_p - \dot{\mathbf{x}}_m)'Q(\dot{\mathbf{x}}_p - \dot{\mathbf{x}}_m) + \mathbf{u}'R\mathbf{u}\} dt \quad (17)$$

where Q and R are weighting matrices and are invariably diagonal. Thus, in the case of the airplane problem this is

$$\min_{\mathbf{u}} \int_{t_0}^{t_f} \{Q_{11}\Delta\dot\beta^2 + Q_{22}\Delta\dot\phi^2 + Q_{33}\Delta\dot P_s^2 + Q_{44}\Delta\dot R_s^2 + R_{11}\delta_R^2 + R_{22}\delta_A^2\} dt \quad (18)$$

where, for example, $\Delta\dot\beta$ indicates the difference between the actual sideslip rate and that which would exist if the model dynamics were substituted.

A procedure which has been found successful for selecting the weighting factors is that "equivalent errors" in $\Delta\dot{\mathbf{x}}$ should give equal contributions to the performance integrand, and the control effort used to counteract these errors should also produce an equal cost contribution.[6] Specifically, the maximum expected errors and maximum available control effort have been weighted to give unit contributions. This condition is satisfied by the criterion

$$\int_{t_0}^{t_f} \left\{\left(\frac{\Delta\dot\beta}{\dot\beta_x}\right)^2 + \left(\frac{\Delta\dot\phi}{\dot\phi_x}\right)^2 + \left(\frac{\Delta\dot P_s}{\dot P_{sx}}\right)^2 + \left(\frac{\Delta\dot R_s}{\dot R_{sx}}\right)^2 + \left(\frac{\delta_R}{\delta_{R_x}}\right)^2 + \left(\frac{\delta_A}{\delta_{A_x}}\right)^2\right\} dt \quad (19)$$

where subscript x indicates the maximum value.

Maximum rate errors have been estimated by considering the response of the model airplane alone. The peak sideslip rate and yaw acceleration experienced in normal operation occur following the most severe side gust, say 50 fps. The maximum $\dot\beta$ is then $\dot\beta_x \approx (50/V_p)/1.7$ rad/sec and $\dot R_{sx} = (50/V_p)N_\beta{}'_{\text{model}}$. The maximum roll rate was taken as 10°/sec, so the maximum roll acceleration is approximately $\dot P_{sx} = L'_p\dot\phi_x = -20°/\text{sec}^2$. Now the maximum state variable errors have been specified and it is postulated that maximum control effort can be used to correct these errors. For this particular airplane these are 1) landing and take-off: $\delta_{R_x} = 25°$, $\delta_{A_x} = 50°$ (spoilers); 2) mach 0.5 thru 1.1: $\delta_{R_x} = 15°$, $\delta_{A_x} = 30°$; 3) above Mach 1.1: $\delta_{R_x} = 10°$, $\delta_{A_x} = 20°$. Substituting these values in Eq. (19) gives a performance criterion appropriate to any flight condition.

Naturally, there is nothing absolute about this criterion and it is probable that changes will be needed at some flight conditions, because of differences in control surface effectiveness for example. However it is a plausible scheme, it does account for scaling of the variables, and in practice only minor adjustments of the weighting factors have been necessary. The control interval $t_f - t_0$ is generally made infinite so that the steady-state solution of the ricatti equation is used and the optimum control law is invariant with time. However, it is not essential to compute the steady-state solution in order to obtain a useful invariant feedback system, since the ricatti matrix P varies only because in the conventional formulation the control interval is shrinking. Thus, if a floating control interval is considered in which the initial time, t_0, moves with the present time, and the time-to-go is always say T sec (20 sec was used here) then the fixed feedback gains corresponding to $P(T)$ are appropriate.

Measurement Transformations

Having specified the control problem and knowing the form of the solution the practical difficulty of taking feedback from all the state variables is evident. One problem is that the rates used in the state variable representation are stability axis rates, and therefore to measure these would require changing the orientation of the gyros with angle of attack. This may be negligible at high speed, but the difference is significant at landing. Of greater importance is the feedback of sideslip. This is not impossible but it is much more difficult than measuring side acceleration. These problems can be avoided by simple transformations.

The state equations are

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{20}$$

and it has been shown that the optimal control law is

$$\mathbf{u} = D\mathbf{x} \tag{21}$$

Now suppose it is preferred to use another vector $\mathbf{y}$ as the feedback vector rather than $\mathbf{x}$, and that $\mathbf{y}$ and $\mathbf{x}$ are equivalent for the closed-loop system and linearly related by

$$\mathbf{y} = F\mathbf{x} + G\mathbf{u} \tag{22}$$

For the closed-loop system

$$\mathbf{y} = (F + GD)\mathbf{x}$$

$$\mathbf{x} = (F + GD)^{-1}\mathbf{y}$$

and the corresponding feedback control law with vector $\mathbf{y}$ is

$$\mathbf{u} = D(F + GD)^{-1}\mathbf{y} \tag{23}$$

This result can be used to transform the optimal controller so that e.g. side acceleration and body axis rates are required for feedback in place of sideslip angle and stability rates. The necessary relationships are

$$P_B = P_s \cos\alpha - R_s \sin\alpha$$

$$R_B = P_s \sin\alpha + R_s \cos\alpha \tag{24}$$

$$A_y = Y_\beta\beta + Y_p P_s + Y_r R_s + Y_{\delta_R}\delta_R + Y_{\delta_A}\delta_A \tag{25}$$

with the other variables being the same in $\mathbf{y}$ and $\mathbf{x}$, i.e., $\mathbf{y} = [A_y,\phi,P_B,R_B,\delta_W,\delta_p]'$. Putting this in the matrix form of Eq. (22) facilitates transformation of the original optimal control law to one employing only easily measurable variables.

Generalized Inverse Matrix in Control System Synthesis

The optimal control techniques discussed heretofore are based on the minimization of an integral performance criterion, but a simple least square minimization is possible in the case of implicit model-following. That is, the error vector

$$\mathbf{e} = A\mathbf{x} + B\mathbf{u} - H\mathbf{x} \tag{26}$$

is formed and the algebraic performance criterion

$$\begin{aligned} L &= \mathbf{e}'\mathbf{e} \\ &= (A\mathbf{x} + B\mathbf{u} - H\mathbf{x})'(A\mathbf{x} + B\mathbf{u} - H\mathbf{x}) \end{aligned} \tag{27}$$

is minimized by ordinary calculus

$$\partial L/\partial \mathbf{u} = 2B'(A\mathbf{x} + B\mathbf{u} - H\mathbf{x}) = 0$$

Hence the optimal control law is

$$\mathbf{u}^0 = -(B'B)^{-1}B'(A - H)\mathbf{x} \tag{28}$$

The matrix product $(B'B)^{-1}B'$ is termed the generalized inverse, $B^\#$, of matrix B[10]. It has certain properties which make it analogous to the ordinary inverse of nonsingular matrices, but it is applicable to singular and nonsquare matrices. Specifically, 1) $AA^\#A = A$, 2) $A^\#AA^\# = A^\#$, 3) $(A^\#A)' = A^\#A$, 4) $(AA^\#)' = AA^\#$. These can easily be proved since $B^\#B = I$, although this is not a necessary condition for a generalized inverse, and $B'B$ is symmetric.

Extending this to include a weighting matrix

$$L = (A\mathbf{x} + B\mathbf{u} - H\mathbf{x})'Q(A\mathbf{x} + B\mathbf{u} - H\mathbf{x})$$

gives

$$\mathbf{u}^0 = -(B'QB)^{-1}B'Q(A - H)\mathbf{x} \tag{29}$$

This control law is the same as that obtained by the implicit model-following when the ricatti matrix P and weighting matrix R are null. It has been indicated in the literature that whenever zero weight is placed on control then the ricatti matrix is null,[3] but this is only the case if perfect matching of plant and model is possible. Briefly, this can be seen by considering the optimal return function

$$V(\mathbf{x},\tau) = \mathbf{x}'P\mathbf{x} = \min_{u(t)} \int_\tau^T L dt \tag{30}$$

Since L is a positive semidefinite function (i.e., $L \geqq 0$) then for any $\mathbf{x}$

$$\mathbf{x}'P\mathbf{x} = \min_{\mathbf{u}(t)} \int_\tau^T L dt = 0 \tag{31}$$

only if $L = 0$ over the entire interval τ to T; that is, perfect matching has been achieved.

When using the simple control law Eq. (29) certain conditions must be satisfied in order to ensure that it exists and is not zero. Specifically, every control variable must enter directly into at least one of the plant equations which it is desired to modify, and the number of nonidentical equations which are weighted and controllable must equal or exceed the number of control variables. A significant consequence of the first condition is that the absence of actuator lags is necessary to use this approach. In rigid body synthesis this is not usually a problem since the actuator dynamics are at least a decade in frequency higher than rigid body motions, but in other problems transformation of the system equations may be required.

Perfect Model-Following

In the case of moderate order systems such as the rigid body lateral-directional control problem, it is feasible to evaluate Eq. (29) by hand and reveal explicitly the relationship between the model, the weighting factors and the feedback gains, a matter of primary concern to the practicing engineer. However, doing this for the complete problem yields very complex expressions and simplifications are desirable. It is found that the effect of weighting the sideslip equation is quite small, because the difference between the plant and model equations is small and also $Y_{\delta_R}{}^*$ and $Y_{\delta_A}{}^*$ are small. Neglecting these terms leaves two controls and two controllable equations to be changed, and therefore perfect matching is possible. The feedback gains then are quite simple, for example

$$\begin{aligned} D_{11} &= \frac{\delta_R}{\beta} = \frac{-L'_{\delta_A}(A_{41} - H_{41}) + N'_{\delta_A}(A_{31} - H_{31})}{L'_{\delta_A}N'_{\delta_R} - L'_{\delta_R}N'_{\delta_A}} \\ D_{21} &= \frac{\delta_A}{\beta} = \frac{+L'_{\delta_R}(A_{41} - H_{41}) - N'_{\delta_R}(A_{31} - H_{31})}{L'_{\delta_A}N'_{\delta_R} - L'_{\delta_R}N'_{\delta_A}} \end{aligned} \tag{32}$$

It is interesting to note that these results amount to a broad generalization of the "equivalent derivative" technique which was one of the first techniques used for augmentation system design.[8] In this technique, desired values of the aerodynamic derivatives are selected, and the feedback gains are calculated to provide the necessary augmentation.

Table 1 Optimal feedback gains by implicit model-following

Flight condition	Ay, ft/sec²	ϕ, deg	P_B, deg/sec	R_B, deg/sec	δ_w, deg	Surface, deg
Landing	2.16	−0.40	−0.07	2.99	−0.35	Rudder
Mach 0.85	0.83	−0.08	0.12	2.10	−0.22	
Mach 1.10	1.30	−0.11	0.00	3.57	−0.31	
Mach 2.70	0.96	−0.05	0.09	4.04	−0.25	
Landing	−2.95	−0.11	−0.62	0.18	0.84	Aileron
Mach 0.85	−0.93	−0.01	−1.02	−0.73	0.87	
Mach 1.10	−2.35	0.00	−1.25	−0.12	0.93	
Mach 2.70	−0.55	0.00	−1.72	0.06	1.01	

For example, to augment the directional stability take

$$N'_{\beta\,\mathrm{desired}} = N'_{\beta\,\mathrm{airframe}} + \left(\frac{\delta_R}{\beta}\right) N'_{\delta_R} \qquad (33)$$

therefore the required feedback gain is

$$\left(\frac{\delta_R}{\beta}\right) = (N'_{\beta\,\mathrm{desired}} - N'_{\beta\,\mathrm{airframe}})/N'_{\delta_R} \qquad (34)$$

This corresponds to the above equation for D_{11} and they are equal if $N'_{\delta_A} = 0$. Moreover, if the control system is modified to include cross-feeds between surfaces to cancel the surface coupling then the equivalent derivative and perfect model-following techniques yield identical sideslip, roll and yaw rate feedback gains. The use of a state variable formulation does facilitate the specification of model dynamics however, and in particular it reveals an important change in the structure of the model coefficient matrix and it includes command inputs.

Optimal Results

These techniques have been applied to the design of lateral-directional augmentation systems for supersonic transport aircraft. When working with a particular airplane specific values must be assigned to parameters k_β and k_ω. It has been found necessary to specify $k_\beta < 1$ to prevent increasing the dihedral effect, k_ω was set to 1.25 corresponding to an increase of 56% in directional stability, and time-to-half-amplitude for the dutch roll was chosen as 2.0 sec. Weighting matrices Q and R were determined initially by the equivalent deviation technique. The closed-loop coefficient matrices $(A - BD)$ were then compared with the model (H), and it was found that better matching of the roll response was desirable at Mach 1.1 (to reduce L'_β) and Mach 2.7 (to augment L'_p). Multiplying Q_{33} by four provided satisfactory results. At landing it was evident from the time response that too much weight had been placed on rudder, and therefore this was reduced by a factor of four. In general adjusting the weighting Q is more effective for improving the response than changing R, except when it is obvious that a surface is under utilized. No attempt was made to produce a set of optimal gains invariant with flight condition.

The resulting optimal feedback gains are listed in Table 1 for four of the flight conditions studied. It will be observed that the gains are moderate to low in all cases. Figure 1 shows the optimal response to step gusts (i.e., a step of 50 fps in side velocity) at landing, transonic and supersonic flight conditions. These demonstrate that the dutch roll is fast and well damped, and the roll due to these severe gusts is easily controllable by the pilot. The response to pilot wheel commands to roll to a bank angle of 20° and hold this in a turn is shown in Fig. 2. Roll rate has the desired exponential form with a time constant in the range 0.55–0.70 sec, and the turns are well coordinated. Pilot evaluation of such optimal systems has always been very satisfactory.

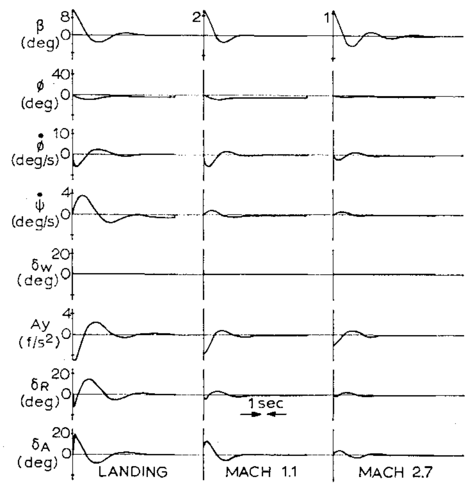

Fig. 1 Step gust responses with optimal control.

If the weight on control is reduced to zero then high feedback gains result (Table 2), and almost perfect matching of the plant and model is achieved. Very similar results are obtained when the simpler generalized inverse technique is employed with this model specification, and therefore to evaluate this approach a less stringent model performance was adopted. Specifically, no augmentation of the dutch roll frequency was required except at supersonic cruise where $k_\omega = 1.1$, the time-to-half amplitude was increased to 2.31 sec, less attenuation of dihedral was called for, and the roll time constant was specified as 0.6 sec. The corresponding control system gains are listed in Table 3. These are in the same range as those obtained by the complete implicit model-following, although they vary more widely due to the rigid nature of the optimization. Also, because near perfect model-following is achieved, the time responses are very close to those of the modified ideal system and are slightly better than those shown in Figs. 1 and 2.

Table 2 Optimal feedback gains with zero weight on control

Flight condition	Ay, ft/sec²	ϕ, deg	P_B, deg/sec	R_B, deg/sec	δ_w, deg	Surface, deg
Landing	5.58	−0.56	0.65	3.89	−0.67	Rudder
Mach 0.85	1.28	−0.09	0.41	2.35	−0.37	
Mach 1.10	4.85	−0.11	0.73	3.25	−0.83	
Mach 2.70	4.81	−0.08	1.16	6.41	−0.92	
Landing	−2.94	−0.10	−0.60	0.03	0.98	Aileron
Mach 0.85	−1.44	0.00	−1.55	−1.48	1.22	
Mach 1.10	−2.91	0.00	−1.40	−0.44	1.07	
Mach 2.70	−1.45	0.00	−2.25	−0.60	1.31	

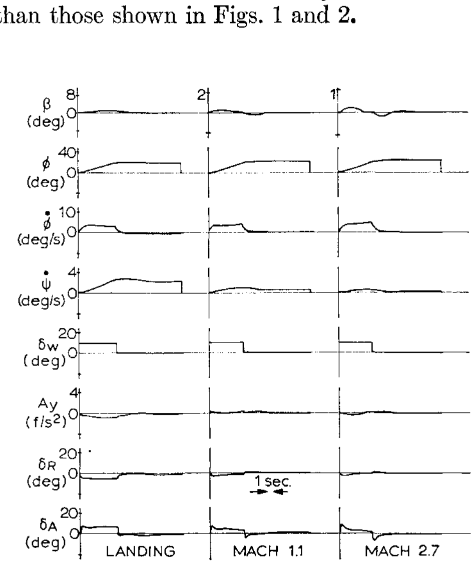

Fig. 2 Wheel input responses with optimal control.

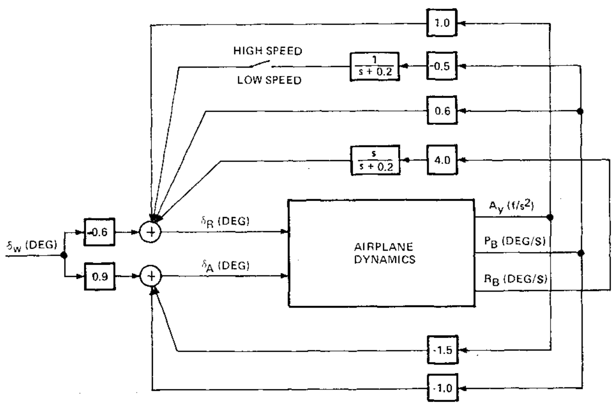

Fig. 3 Suboptimal augmentation system.

Thus good responses and acceptable gains can be obtained by the generalized inverse approach with a significant reduction in computational effort from the integral optimization methods. The greater sensitivity of the gains to plant parameter changes is a disadvantage, and also some flexibility has been lost by not specifying weight on control, although this may be regarded as a virtue rather than a deficiency in that it reduces the number of parameters to be chosen more or less arbitrarily. Nevertheless, the directness of this approach, the trivial computational requirements, and the good response make this method very attractive in the initial stages of stability augmentation system design.

Invariant Suboptimal Systems

Practical suboptimal augmentation systems have been synthesized by taking an approximate average of the optimal results, having regard to the desirability of low gains in order to avoid stability problems with nonlinearities and higher order modes. The results of both implicit model-following and the generalized inverse approach have been used, the primary difference being that the latter technique indicates definite need for roll rate to rudder feedback. The inclusion of this loop does give slightly better responses, and it will be considered here.

Optimal side acceleration to rudder gains lie in the range 0.4–2.9 degree per ft/sec² with the higher values occurring at flight conditions where dihedral effect is great, and values below 1.4 being satisfactory elsewhere. This feedback loop is the one most likely to excite the body bending modes; therefore, this gain must be as small as possible consistent with engine failure and other directional stability objectives. It has been estimated that with normal rudder effectiveness

Table 3 Feedback gains by generalized inverse approach with revised model

Flight condition	Ay, ft/sec²	ϕ, deg	P_B, deg/sec	R_B, deg/sec	δ_w, deg	Surface, deg
Landing	2.87	−0.60	0.89	4.20	−0.73	Rudder
Mach 0.85	0.38	−0.09	0.32	2.29	−0.27	
Mach 1.10	1.94	−0.12	0.81	3.45	−0.66	
Mach 2.70	1.37	−0.08	1.10	6.32	−0.75	
Landing	−2.90	−0.08	−0.26	−0.06	0.73	Aileron
Mach 0.85	−1.26	0.02	−1.15	−1.44	0.92	
Mach 1.10	−2.37	0.01	−1.11	−0.52	0.81	
Mach 2.70	−0.14	0.01	−1.88	−0.85	1.01	

a value of 1.5 is the maximum permissible, and 1.0 was taken as the initial choice. Bank angle feedback is very small at all conditions except landing and take-off, but in these cases it is significant in improving turn coordination (it arises from the $N'_rY_\phi\Phi$ term introduced into the model yaw rate equation). Accordingly, a loop with a gain of −0.5 is switched in for the low-speed flight conditions. Roll rate feedback to rudder is significant in the generalized inverse results, and a value of 0.6 was taken. Yaw rate to rudder feedback is the primary dutch roll damping term, and a gain of 4.0 was adopted. Other averaged values of the feedback gains were: $\delta_R/\delta_W = -0.6$, $\delta_A/A_y = -1.5$ (note that this feedback must be made sufficiently small so that it never produces negative dihedral effect), $\delta_A/\phi = 0.0$, $\delta_A/P_B = -1.0$ $\delta_A/R_B = 0.0$, $\delta_A/\delta_W = 0.9$.

A problem encountered in the design of fixed gain controllers is that a fixed yaw rate feedback gain is incompatible with the specification that the spiral mode should be almost neutrally stable at all flight conditions. The reason for this is that the steady-state rudder resulting from this feedback tends to be greater than is required for a coordinated turn and the resulting sideslip causes the airplane to roll to zero bank angle. (In the optimal controllers the gain on yaw rate is closely related to speed, and also this feedback is offset by bank angle feedback.) This problem was solved by "washing out" the yaw rate signal with a simple high-pass filter so that there is zero steady-state contribution; this solution is in fact standard practice on present day yaw dampers. Similarly, the bank angle feedback used at low speed must be washed out to prevent steady-state feedback, and this can conveniently be done by pseudo-integrating roll rate with a lag network.

A diagram of the final system is given in Fig. 3, and the gust and wheel responses are shown in Figs. 4 and 5. Comparison of the optimal and suboptimal results indicates that the deterioration in the suboptimal cases is quite small,

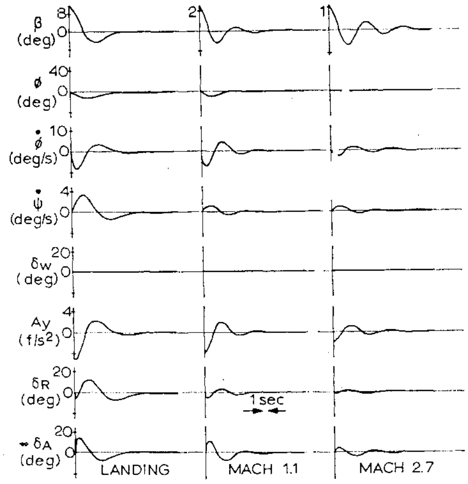

Fig. 4 Step gust responses with suboptimal control.

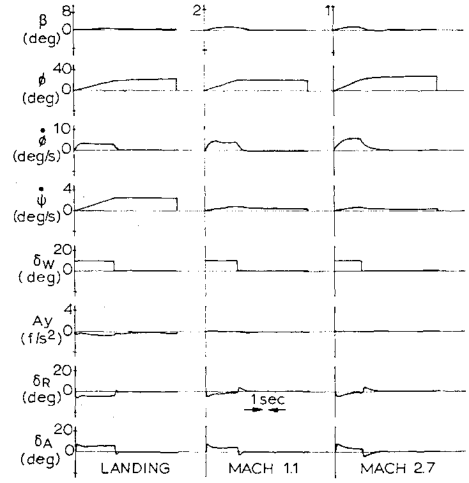

Fig. 5 Wheel input responses with suboptimal control.

and although in general the speed of response and damping is lower with the suboptimal system pilot evaluation has been at least satisfactory.

Conclusions

Any of the optimal model-following techniques presented here can be applied to the design of stability augmentation systems, but in practice the implicit model-following approach is preferred since it yields lower gains and does not require the synthesis of a model system or prefilter. The variational type of optimization and the generalized inverse approach give similar results, and the latter has a significant advantage in computational requirements.

It has been demonstrated that for this airplane, a supersonic transport, it is possible to design a suboptimal, fixed gain augmentation system on the basis of optimal results. This fact has also been observed on several other airplanes, but it cannot be guaranteed in every case since insensitivity to parameter variations is not an explicit objective in the optimal control technique. The use of high-pass filters on bank angle and yaw rate feedback contributes substantially to the feasibility of invariant systems however, and this together with the measurement transformations enables the engineer to obtain practical systems from the optimal results.

The presence of higher order dynamics has been expressly neglected in this study. Such an approach is appropriate and satisfactory in the early stages of airplane design when actuator and structural data is not available and the aerodynamic data is subject to change. Nevertheless, the effect of these dynamics must be analysed at the earliest opportunity, and in general a low-pass filter will be required to prevent excitation of the flexible modes. The use of low gains and careful location of sensors permit a fairly high cut-off frequency, and current experience indicates that the rigid body handling qualities are not significantly deteriorated by the addition of a flexible mode filter. However, as aircraft become more flexible and more sophisticated performance criteria are considered, for example ride qualities improvement, load alleviation and modal suppression, active control of the bending modes will be necessary. This will involve extension of the methods and equations used herein, with this approach providing the basis for the rigid body section of the dynamics, and it is the subject of further research.

Appendix: Numerical Data

The matrices corresponding to the optimal implicit model-following example described in this paper are presented below for the Mach 2.7 flight condition:

$$A = \begin{bmatrix} -3.72E-2, & 1.23E-2, & 5.49E-4, & -1.00E-0, & 0 \\ 0 & ,0 & ,1 & ,0 & ,0 \\ -6.37E-0, & 0 & ,-2.31E-1, & 6.18E-2, & 0 \\ 1.25E-0, & 0 & ,1.60E-2, & -4.57E-2, & 0 \\ 0 & ,0 & ,0 & ,0 & ,0 \end{bmatrix}$$

$$B = \begin{bmatrix} 8.39E-4, & 2.36E-4 \\ 0 & ,0 \\ 8.02E-2, & 8.04E-1 \\ -8.62E-2, & -6.65E-2 \\ 0 & ,0 \end{bmatrix}$$

$$H = \begin{bmatrix} -3.72E-2, & 1.23E-2, & 0 & ,-1.00E-0, & 0 & \\ 0 & ,0 & ,1 & ,0 & ,0 & \\ -4.77E-0, & 0 & ,-2.00E-0, & 0 & ,1.00E-0 & \\ 1.91E-0, & 8.05E-3, & 0 & ,-6.56E-1, & 0 & \\ 0 & ,0 & ,0 & ,0 & ,0 & \end{bmatrix}$$

$$Q = \text{diag}[6.01E+1,\ 1.00E-0,\ 2.50E-1,\ 5.73E-0,\ 0]$$

$$R = \text{diag}[2.50E-1,\ 6.25E-2]$$

$$D = \begin{bmatrix} -1.68E-0, & -5.34E-2, & 4.32E-1, & 4.18E-0, & -2.45E-1 \\ 9.56E-1, & -1.65E-3, & -1.72E-0, & 1.02E-1, & 1.00E-0 \end{bmatrix}$$

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